# Theorem of Inverse Transformation.

Consider a transformation continuously differentiable

$$T: D' \subset \mathbb{R}^2 \longrightarrow D \subset \mathbb{R}^2$$

$$(\xi, \eta) \longrightarrow (x, y) = (x(\xi,\eta), y(\xi,\eta)).$$

And $(\xi_0, \eta_0) \in D'$ such that

$$(x_0, y_0) = (x(\xi_0,\eta_0), y(\xi_0,\eta_0)).$$

If the Jacobian of this transformation $T$

$$J(\xi_0, \eta_0) = \begin{vmatrix} \frac{\partial x}{\partial \xi}(\xi_0,\eta_0) & \frac{\partial x}{\partial \eta}(\xi_0,\eta_0) \\ \frac{\partial y}{\partial \xi}(\xi_0,\eta_0) & \frac{\partial y}{\partial \eta}(\xi_0,\eta_0) \end{vmatrix} \neq 0$$

Then, there are neighborhoods of the points $B(x_0,y_0) \subset D$ and $B'(\xi_0,\eta_0) \in D'$, respectively, and functions $F(x,y)$ and $G(x,y)$ such that to any $(x,y) \in B$ corresponds a unique $(\xi,\eta) \in B'$ such that $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$, and this unique pair is given by

$$\xi = F(x,y) \text{ and } \eta = G(x,y).$$

In other words, the transformation

$$T: D' \longrightarrow D$$
$$(\xi,\eta) \rightarrow (x,y)$$

is one-to-one in a neighborhood of $(\xi_0,\eta_0)$

if $\quad J(\xi_0,\eta_0) = \left(x_\xi y_\eta - x_\eta y_\xi\right)(\xi_0,\eta_0) \neq 0.$

Remark: If $\quad T: B' \rightarrow B$ continuously diff. $(T \in C^1)$

is one-to-one and onto, then $T$ is called

a Diffeomorphism.

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We noticed that Amsden-Hirt grid generator:

$$T: D' \longrightarrow D$$
$$(\xi,\eta) \longrightarrow (x,y) = (x(\xi,\eta), y(\xi,\eta))$$

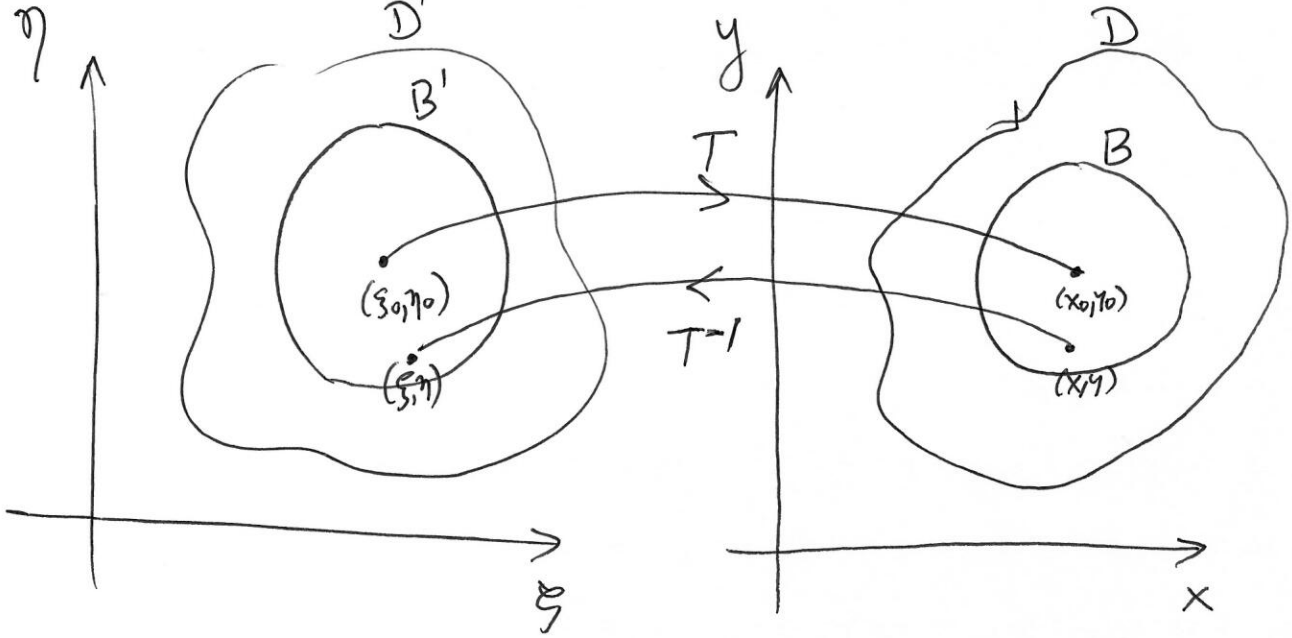
given by

$$\nabla^2_{\xi,\eta} x = x_{\xi\xi} + x_{\eta\eta} = 0$$
$$\nabla^2_{\xi,\eta} y = y_{\xi\xi} + y_{\eta\eta} = 0 \qquad + \text{ BCs.}$$

produce folded grids for non-convex regions.

It means there are points $(x_0,y_0) = (x(\xi_0,\eta_0), y(\xi_0,\eta_0))$

where $\quad J(\xi_0,\eta_0) = 0.$




$$T: D' \longrightarrow D$$

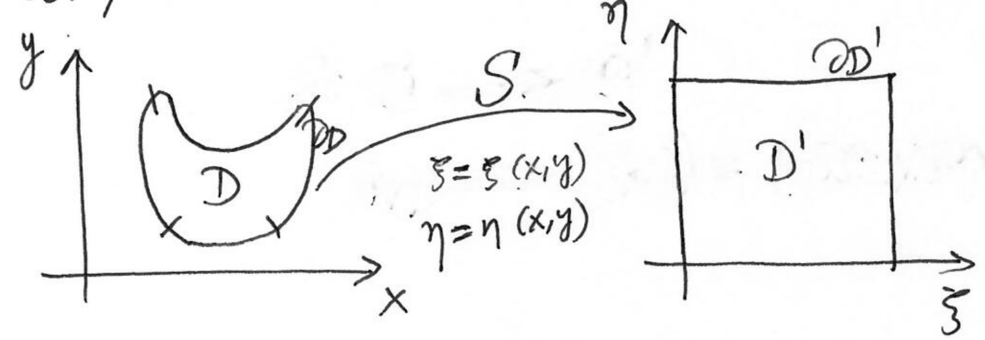
$$(\xi, \eta) \longrightarrow (x, y)$$

is locally one-to-one in a neighborhood of $(\xi_0, \eta_0)$.

if $J(\xi_0, \eta_0) \neq 0$.

# Winslow or Smoothness Grid Generation

Consider a transformation $S$ from a physical domain $D$ to a computational domain $D'$ (rectangular)




First, consider the boundary transformation:

$$\partial S : \partial D \longrightarrow \partial D'$$
$$(x_b, y_b) \longrightarrow (\xi_b, \eta_b). \quad (3.1)$$

Secondly, consider the transformation of the interior points

$$S : D \longrightarrow D'$$
$$(x,y) \longrightarrow (\xi, \eta) = (\xi(x,y), \eta(x,y)).$$

defined by

$$\begin{cases} \xi_{xx} + \xi_{yy} = 0 \\ \eta_{xx} + \eta_{yy} = 0 \end{cases} \quad (3.2)$$

Since the transformed region $D'$ (rectangle) is Convex and the transformation $S$ Satisfies a maximum principle, then $S(D) \subseteq D'$

This Suggests the following idea to Generate a non-folded grid in the physical domain. $D$:

Find conditions on $D$, $D'$ and $\partial s$ Such that $T = S^{-1}$ exists

$$T : D' \longrightarrow D$$

Since this transformation $T$ would be one-to-one and onto, any grid generated in $D$ from this transformation will not be folded.

In practice, we will consider a uniform rectangular Grid in $D'$ $(\xi$-$\eta)$ and the desired grid in $D$ Will be the image of this rectangular grid by $T = S^{-1}$.

Rado's Theorem guarantee the existence of $T = S^{-1}$ under certain conditions.

Thm. (Rado)

For the transformation $S$ defined by (3.1) and (3.2) if

i) $D$ and $D'$ are Simply Connected and bounded, so $\partial D$ and $\partial D'$ are simple closed curves.

ii) $\partial S : \partial D \to \partial D'$ is a homeomorphism

iii) $D'$ is a convex domain

then,

the Jacobian of the transformation $S$

$J(x,y) \neq 0$, for all $(x,y) \in D$

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Corollary.

Under the assumptions of Rado's Thm the transformation (3.1) and (3.2) has an inverse defined by

$$\partial T = \partial S^{-1} : \partial D' \longrightarrow \partial D.$$
$$(\xi_b, \eta_b) \longrightarrow (x_b, y_b)$$

and
$$T : D' \longrightarrow D$$
$$(\xi, \eta) \longrightarrow (x, y) = (x(\xi,\eta), y(\xi,\eta))$$

given by the winslow's quasi-linear elliptic system:

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \qquad (\xi,\eta) \in D'$$
$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0$$

where
$$\alpha = x_\eta^2 + y_\eta^2, \quad \beta = x_\eta x_\xi + y_\xi y_\eta$$
$$\gamma = x_\xi^2 + y_\xi^2.$$

<u>Proof.</u> Due to inverse theorem

$$x = x(\xi,\eta), \qquad y = y(\xi,\eta)$$

or

$$x - x(\xi,\eta) = 0, \qquad y - y(\xi,\eta) = 0$$

or

$$\begin{cases} x - x\left(\xi(x,y), \eta(x,y)\right) = 0 & (2.1) \\ y - y\left(\xi(x,y), \eta(x,y)\right) = 0 & (2.2) \end{cases}$$

Differentiating (2.1)-(2.2) w.r. to $x$

$$1 - x_\xi\left(\xi(x,y), \eta(x,y)\right)\xi_x - x_\eta\left(\xi(), \eta()\right)\eta_x = 0 \quad (2.3)$$

$$0 - y_\xi\left(\xi(), \eta()\right)\xi_x - y_\eta\left(\xi(), \eta()\right)\eta_x = 0 \quad (2.4)$$

In matrix form

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (2.5)$$

Analogously, differentiating (2.1)-(2.2) w.r.t. $y$

$$\begin{cases} 0 - x_\xi()\xi_y - x_\eta()\eta_y = 0 \\ 1 - y_\xi()\xi_y - y_\eta()\eta_y = 0 \end{cases}$$

In matrix form

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_y \\ \eta_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (2.6)$$

Then, Combining (2.5) and (2.6)

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.1)$$

Therefore,

if $J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$ then $J^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$

but

$$J^{-1} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}, \quad \text{Since } J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0.$$

As a consequence,

$$\frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

and

$$\boxed{\begin{aligned} \xi_x &= \frac{y_\eta}{J}, & \xi_y &= -\frac{x_\eta}{J} \\ \eta_x &= \frac{-y_\xi}{J}, & \eta_y &= \frac{x_\xi}{J}. \end{aligned}} \qquad (3.2)$$

First order derivatives of $\xi(x,y)$ and $\eta(x,y)$ in terms of first order derivatives of $x(\xi,\eta)$ and $y(\xi,\eta)$.

Now, we want to obtain <u>Second order derivatives of $\xi(x,y)$ and $\eta(x,y)$</u> in terms of <u>Second order and first order derivatives of $x(\xi,\eta)$ and $y(\xi,\eta)$</u>.

Differentiating (2.3) and (2.4) again w.r.t. $x$

$$\begin{cases} x_{\xi\xi}\xi_x^2 + x_{\xi\eta}\xi_x\eta_x + x_\xi\xi_{xx} + x_{\eta\xi}\eta_x\xi_x + x_{\eta\eta}\eta_x^2 + x_\eta\eta_{xx} = 0 \\ y_{\xi\xi}\xi_x^2 + 2y_{\xi\eta}\xi_x\eta_x + y_\xi\xi_{xx} + y_{\eta\eta}\eta_x^2 + y_\eta\eta_{xx} = 0 \end{cases}$$

From these equations the following system for $\xi_{xx}$ and $\eta_{xx}$ is obtained

$$x_\xi\xi_{xx} + x_\eta\eta_{xx} = -\left(x_{\xi\xi}\xi_x^2 + 2x_{\xi\eta}\xi_x\eta_x + x_{\eta\eta}\eta_x^2\right) \overset{\text{using (3.2}}{=\!=} -\frac{1}{J^2}\left(x_{\xi\xi}y_\eta^2 - 2x_{\xi\eta}y_\xi y_\eta + x_{\eta\eta}y_\xi^2\right) \quad (4.1)$$

$$y_\xi\xi_{xx} + y_\eta\eta_{xx} \overset{\text{using (3.2)}}{=\!=} -\frac{1}{J^2}\left(y_{\xi\xi}y_\eta^2 - 2y_{\xi\eta}y_\xi y_\eta + y_{\eta\eta}y_\xi^2\right) \quad (4.2)$$

Let's define

$$A = -\left(x_{\xi\xi}y_\eta^2 - 2x_{\xi\eta}y_\xi y_\eta + x_{\eta\eta}y_\xi^2\right)$$

$$B = -\left(y_{\xi\xi}y_\eta^2 - 2y_{\xi\eta}y_\xi y_\eta + y_{\eta\eta}y_\xi^2\right)$$

Thus, (4.1) and (4.2) can be written as

$$\begin{cases} x_\xi \, \xi_{xx} + x_\eta \, \eta_{xx} = \frac{1}{J^2} A & (5.1) \\ y_\xi \, \xi_{xx} + y_\eta \, \eta_{xx} = \frac{1}{J^2} B & (5.2) \end{cases}$$

Proceeding in an analogous form, we can differentiate (2.3) and (2.4) w.r.t. "$y$", and obtain the system

$$\begin{cases} x_\xi \, \xi_{yy} + x_\eta \, \eta_{yy} = \frac{1}{J^2} C & (5.3) \\ y_\xi \, \xi_{yy} + y_\eta \, \eta_{yy} = \frac{1}{J^2} D & (5.4) \end{cases}$$

where

$$C \equiv -\left(x_{\xi\xi} x_\eta^2 - 2 x_{\xi\eta} x_\xi x_\eta + x_{\eta\eta} x_\xi^2\right)$$

$$D \equiv -\left(y_{\xi\xi} x_\eta^2 - 2 y_{\xi\eta} x_\xi x_\eta + y_{\eta\eta} x_\xi^2\right)$$

From here, there is only two more steps to prove the theorem.

1) a) Solve system (5.1)-(5.2) for $\xi_{xx}$ and $\eta_{xx}$.

   b) Solve system (5.3)-(5.4) for $\xi_{yy}$ and $\eta_{yy}$.

2) Substitute $\xi_{xx}, \eta_{xx}, \xi_{yy}, \eta_{yy}$ obtained in (1) into the system

$$\begin{cases} \xi_{xx} + \xi_{yy} = 0 \\ \eta_{xx} + \eta_{yy} = 0 \end{cases}$$

and use that $J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0$ to obtain the Winslow system of equations.

$$\begin{cases} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \end{cases}$$

$$\alpha \equiv x_\eta^2 + y_\eta^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi^2 + y_\xi^2.$$

So the transformation $T$ sought is defined as

$$T: D' \longrightarrow D$$
$$(\xi,\eta) \rightarrow (x,y) = (x(\xi,\eta),\, y(\xi,\eta))$$

where $x(\xi,\eta)$ and $y(\xi,\eta)$ are the solutions of the system of the following partial differential equations with Dirichlet BCs defined on the boundary of $D'$.

$$\begin{cases} \alpha\, x_{\xi\xi} + 2\beta\, x_{\xi\eta} + \gamma\, x_{\eta\eta} = 0 & (*) \\ \alpha\, y_{\xi\xi} - 2\beta\, y_{\xi\eta} + \gamma\, y_{\eta\eta} = 0 & (**) \end{cases}$$

where $\alpha = x_\eta^2 + y_\eta^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi^2 + y_\xi^2.$

The system (*)-(**) is

a) Coupled (coefficients)

b) Quasilinear

c) Elliptic.

For the discretization of Winslow's equations,

Use 2nd order centered difference with $\Delta\xi = 1 = \Delta\eta$.

It means

$$\xi_i = i, \quad i = 2, .., N_1 - 1, \quad \Delta\xi = 1. \qquad (*)$$

$$\eta_j = j, \quad j = 2, .., N_2 - 1 \quad \Delta\eta = 1$$

and show that

$$x_{ij} = \frac{1}{2(\alpha_{ij} + \gamma_{ij})} \Big( \alpha_{ij} (x_{i+1,j} + x_{i-1,j}) + \gamma_{ij} (x_{i,j+1} + x_{i,j-1})$$

$$- \frac{1}{2} \beta_{ij} \left( x_{i+1,j+1} - x_{i+1,j-1} - x_{i-1,j+1} + x_{i-1,j-1} \right) \Big) \qquad (T9.1)$$

where

$$\alpha_{ij} = (x_\eta)_{ij}^2 + (y_\eta)_{ij}^2, \quad \gamma_{ij} = (x_\xi)_{ij}^2 + (y_\xi)_{ij}^2$$

and $\beta_{ij} = (x_\xi)_{ij} (x_\eta)_{ij} + (y_\xi)_{ij} (y_\eta)_{ij}$

Use second order centered difference to approximate <u>all these 1st order derivatives.</u>

Remark (*): The boundary point

$x(1,j)$ and $x(N_1,j)$, $x(i,1)$, $x(i,N_2)$

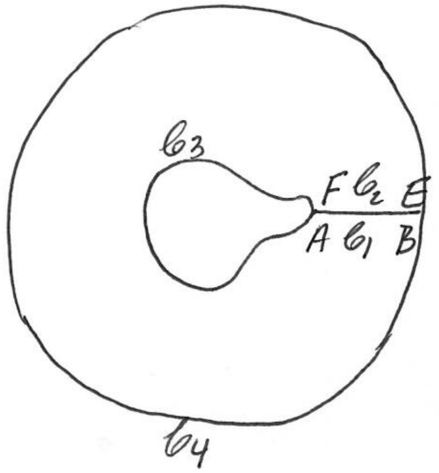
are fixed from BCs. They never change.

The Approximate Solution of (T9.1) and the analogous equ. for the y coordinate, although nonlinear, can be approximated using an iterative technique such as SOR. So, we will proceed analogous as we did with the System of Laplace equations of Amsden-Hirt.

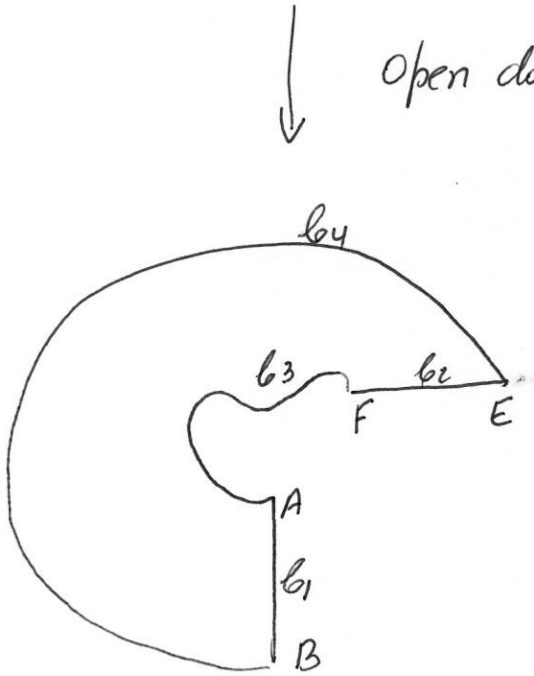
Rado's theorem guarantee the existence of Solutions for these non-linear System which is the inverse of a System of two Laplace equations for the x and y coordinates, respectively.

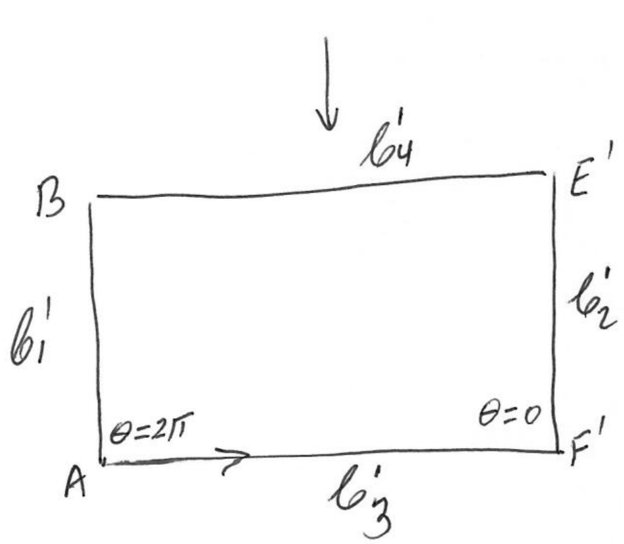
However, there is not a known theorem on how to choose $\omega$ for optimum convergence, or even for convergence. Thus, a trial and error may lead to an appropriate $\omega$. It's always advisable to start with those $\omega$ that work for Amsden-Hirt.

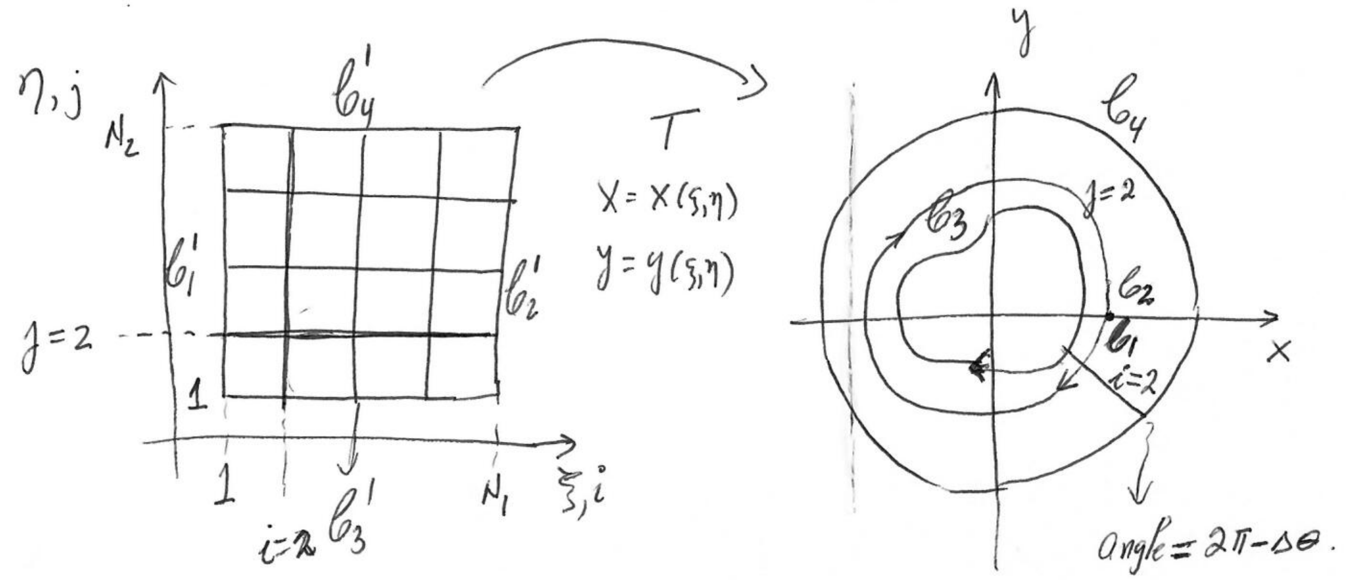
# Multiply Connected Domains




Open domain Clockwise.

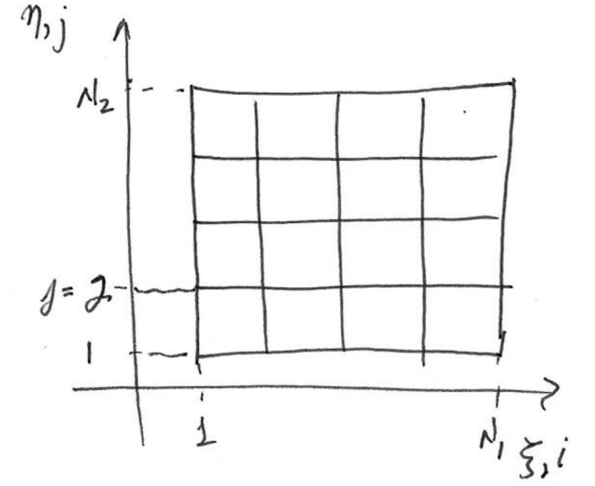





Discuss creation of initial grid.

# Grid Generation Proceedure




While (Iterate > TOL)

$K_1 = K + 1$

for $j = 2 : N_2 - 1$

do computations for special case

branch cut $i = 1$ ($i = N_1$.) (T14.1) (next page.)

Then do the rest

for $i = 2, N_1 - 1$.

(T9.1).

end

end

end

$$X_{1,j} = \frac{1}{2(\alpha_{1j} + \gamma_{1j})} \Big( \alpha_{1j} \left( X_{2,j} - X_{N_1-1,j} \right) + \gamma_{1,j} \left( X_{1,j+1} - X_{1,j-1} \right)$$

$$- \frac{1}{2} \beta_{1,j} \left( X_{2,j} - X_{2,j-1} - X_{N_1-1,j+1} + X_{N_1-1,j-1} \right) \Big) \qquad (T14.1)$$

Similarly, the first derivatives at $i=1$ are discretized.